# Chapter 7: Time Complexity

# Time complexity

Let *M* be a deterministic Turing machine that halts on all inputs. The running time or *time complexity* of *M* is the function f: $N \rightarrow N$, where f(n) is the maximum number of steps that *M* uses on any input of length n. If f(n) is the running time of *M,* we say that

- *M* runs in time f(n)
- *M* is an f(n) time Turing machine

# Big-O notation

- Let f and g be functions $f, g: N \to R^+$. $f(n)=O(g(n))$ if positive integers c and $n_0$ exist such that for every integer $n \geq n_0$

  $$f(n) \leq cg(n)$$

  g(n) is an asymptotic upper bound for f(n).

# Small-o notation

- Let f and g be functions f, g: $N \to R^+$.

$f(n) = o(g(n))$ if for every positive integer c, $n_0$ exists such that for every integer $n \geq n_0$

$$f(n) < cg(n)$$

# Time complexity class

TIME(t(n)) = {L| ∃ TM M s.t.

1) L(M) = L, and

2) ∀ input w, M halts on w in O(t(n)) steps (language L is decided by an O(t(n)) time Turing machine)}

# The class **P**

**P** is the class of languages that are decidable in polynomial time on a deterministic single-tape Turing machine. In other words,

$$P = \bigcup_k TIME(n^k)$$

# Example 1: A language known to be in **P**

$$\text{CONN} = \{\langle G\rangle \mid G \text{ is a connected graph}\}$$

# Example 2: A language known to be in **P**

PATH ={<G, s, t>| G is a graph, and s, t are two vertices. There is a path from s to t.}

# PATH is in P

A polynomial time algorithm *M* for PATH operates as follows:

M = "On input <G, s, t>, where *G* is a directed graph with nodes *s* and *t*:

1. Place a mark on node *s*.
2. Repeat the following until no additional nodes are marked:
3. Scan all the edges of G. If an edge (a, b) is found going from a marked node *a* to an unmarked node *b*, mark node b.
4. If *t* is marked, *accept*. Otherwise, *reject*."

# Example 3: A language known to be in **P**

Every context-free language is in P

# Every CFL is in P

Let G be a CFG in CNF generating the CFL L. Assume that S is the start variable. A polynomial time algorithm *D* works as follows:

*D* = " On input $w = w_1 \dots w_n$:

1. If w = ε and S → ε is a rule, accept.
2. For i = 1 to n:
3. For each variable A:
4. Test whether A → b is a rule, where $b=w_i$.
5. If so, place A in table (i, i).
6. For l = 2 to n:
7. For i = 1 to n-l+1:
8. Let j = i+l-1,
9. For k = i to j-1:
10. For each rule A →BC:
11. If table(i, k) contains B and table(k+1, j) contains C, put A in table(i, j).
12. If *S* is in table(1, n), *accept*. Otherwise, *reject*. "

# Verifier

A **verifier** for a language A is an algorithm V, where

$A = \{w \mid V \text{ accepts } \langle w, c\rangle \text{ for some string } c\}$.

The time of a verifier is measured in terms of the length of w: |w|.

A **polynomial time verifier** runs in a polynomial time in the length of w. A language A is **polynomially verifiable** if it has a polynomial time verifier.

# The class NP, definition

**NP** is the class of languages that have polynomial time verifiers.

HAMPATH = {<G, s, t>| G is a directed graph with a Hamiltonian path from s to t} is in NP

Proof: the following is a NTM that decides HAMPATH problem in nondeterministic polynomial time.

# HAMPATH={<G, s, t>| G is a directed graph with a Hamiltonian path from s to t} is in NP

N="On input <G, s, t>, where G is a directed graph with nodes s and t:

1. Write a list of m numbers, $p_1, \ldots, p_m$, where m is the number of nodes in G. Each number in the list is nondeterministically selected to be between 1 and m.
2. Check for repetitions in the list. If any are found, reject.
3. Check whether s= $p_1$ and t= $p_m$ . If either fail, reject.
4. For each i between 1 and m-1, check whether $(p_i, p_{i+1})$ is an edge of G. If any are not, reject. Otherwise, all tests have been passed, so accept."

# Theorem

Language L is in NP iff it is decided by some nondeterministic polynomial Time Turing machine.

# Proof

($\longrightarrow$) Let A $\in$ NP and A is decided by a polynomial time NTM *N*. Let *V* be the polynomial time verifier for A that exists by the definition of NP. Assume that *V* is a TM that runs in time $n^k$ and construct *N* as follows:

N= "On input *w* of length n:

1. Nondeterministically select string *c* of length at most $n^k$
2. Run V on input (w, c).
3. If V accepts, accept; otherwise, reject.

# Proof

( $\longleftarrow$ ) Assume that A is decided by a polynomial time NTM N and constructs a polynomial time verifier V as follows:

N= “On input <*w*, *c*>, where w and c are strings:

1. Simulate N on input w, treating each symbol of c as a description of the nondeterministic choice to make at each step.
2. If this branch of N’s computation accepts, accept; otherwise, reject.”

# Time classes: **NTIME**(f)

**NTIME**(t(n)) = {L| L is a language decided by an O(t(n)) time nondeterministic Turing machine}

# The class **NP**

$$NP = \bigcup_{k} NTIME(n^k)$$

# Languages in NP

CLIQUE, SUBSET-SUM, SAT, 3SAT, FACTOR, ISO, VERTEX-COVER

SAT = {$\phi$:  $\phi$ is a boolean formula that has a satisfying assignment}

<u>Polynomial verifier</u>:  guess assignment to variables, plug into $\phi$, then verify

# FACTOR= {(k, N): N has a non-trivial factor less than k}

Polynomial verifier:

- guess 1<d<k
- divide d by N
- determine whether the remainder is 0

# Polynomial time computable function

- A function *f* is a polynomial time computable function if some polynomial time Turing machine *M* exists that halts with just *f(w)* on its tape, when started on any input *w*.

# Polynomial-time Reduction

A function $\mathbf{f} : \Sigma^* \rightarrow \Sigma^*$ is a polynomial-time reduction from language $L_1$ to language $L_2$, $\mathbf{L_1 \le_p L_2}$, iff

- $\mathbf{f}$ is a polynomial time computable function.
- for all w,

$$w \in L_1 \leftrightarrow \mathbf{f}(w) \in L_2$$

# Theorem: If $A \leq_p B$ and $B \in P$, then $A \in P$

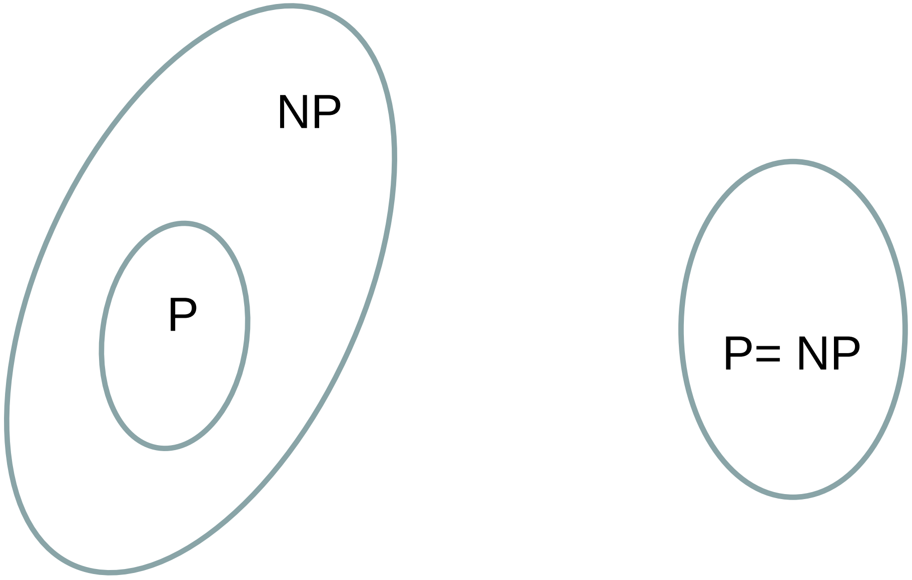
Proof:

Let M be the polynomial time algorithm deciding B and f be the polynomial time reduction from A to B. We describe a polynomial time algorithm N that decides A as follows.

N = “On input w:

1. Compute f(w).
2. Run M on input f(w) and output whatever M outputs.”

# P vs. NP



# NP-hard

A language L is **NP-hard**

iff

$\forall A \in NP, \; A \leq_p L$

# NP-complete

A language L is **NP-complete**

iff

1) $L \in NP$

2) L is NP-hard

# The Cook-Levin Theorem:
# SAT is NP-complete

$\text{SAT} \in \text{P}$ iff $\text{P} = \text{NP}$

# 3-SAT

$$\text{3-SAT} = \{\phi(x_1, x_2 \ldots x_n) \mid \phi \in \text{SAT},$$
$$\phi = C_1 \textbf{ AND } C_2 \textbf{ AND } \ldots C_n$$
$$\text{where } C_i = \alpha_{i1} \textbf{ OR } \alpha_{i2} \textbf{ OR } \alpha_{i3},$$
$$\text{and } \forall i,j\ \exists k\text{:}\ \ \alpha_{ij} = x_k \text{ or } \alpha_{ij} = \underline{x_k}\ \}$$

*<u>Conjunctive Normal Form</u>*

# 3-SAT is NP-complete

Proof:

1) **3-SAT $\in$ NP** (by guessing the assignment to variables and verifying that $\phi(x_1, x_2 \ldots x_n) = 1$)

2) $\forall$**L, L $\in$ NP L $\leq_p$ 3-SAT**

# Vertex Cover

Given graph $G = (V,E)$, a subset $V_1$ of vertices is a vertex cover if each edge in $E$ is adjacent to at least one vertex in $V_1$

Vertex-Cover = {(G,k)| G has a vertex cover of size at most k}

# Directed Hamiltonian Path

**Directed_HP** = {(G,s,t)| G is a directed graph with a path that starts at vertex s, ends at vertex t, and visits every vertex of G <u>exactly once</u>}

# CLIQUE

**CLIQUE** = {{(G, k)| $\exists$ S, S $\subseteq$ $V_{G,}$ s.t. |S| = k, and S is fully connected inside G}

# SUBSET_SUM

**SUBSET_SUM** = $\{(A,k) \mid$ A is a set (list) of integers, s.t. $\exists B: B \subseteq A$ where $\Sigma_{w \in B} w = k\}$